From Multiloop Spinor Helicity Technique to String Reorganization [†]

C.S. Lam*

Department of Physics, McGill University
3600 University St., Montreal, P.Q., Canada H3A 2T8

ABSTRACT

The success of the spinor helicity technique for tree processes is reviewed. To apply it to multiloop diagrams one is naturally led to the Schwinger proper-time representation, whose properties are discussed. This representation also serves as a useful link between field theories and string theories.

1. Introduction

It is well known that the tree-level cross-section σ_n for producing n *soft* photons in the process $e^+ + e^- \to \mu^+ + \mu^- + \gamma_1 + \ldots + \gamma_n$ factorizes, $\sigma_n = \sigma_0 \prod_{i=1}^n F(k_i)$, with σ_0 being the $e^+ + e^- \to \mu^+ + \mu^-$ annihilation cross-section and k_i the momentum of the ith photon. What had not been known until recently[1,2] was that such a factorization remained valid for *hard* photons all carrying the same helicity. The *spinor helicity technique* was invented[1] to give an easy proof in the case $n = 1$. Subsequent refinement[2] of the technique made it possible to prove the factorization for a general n, and to apply it to QCD calculations. Since then, the technique had been used successfully to calculate many *tree* processes impossible or too difficult to do by the usual means. A good example of this is the Parke-Taylor formula[3] so obtained, which gives a simple and exact expression for the pure gluonic process $g + g \to g_1 + \cdots g_n$ for n produced gluons all carry the same helicity. For a review of the subject, see Ref. 4.

This technique as discussed could not be used on higher-order processes owing to the presence of loop momenta. This is unfortunate because loop processes are becoming increasingly important in particle physics. Happily this difficulty has now been overcome. The first breakthrough appeared in the one-loop n-gluon process, where superstring techniques[5] were used to bypass the loop-momentum problem. Subsequently it was realized that a first-quantized formalism could lead to the same result as well[6]. For arbitrary multiloop processes, these approaches are ineffective but a purely field-theoretical technique using the Schwinger proper-time representation can be devised[7]. In the special case of one-loop pure gluonic processes discussed above, all approaches give identical results.

† Talk given at the Workshop on the Advances in QCD, Minneapolis, Feb. 18–20, 1994.

* email address: Lam@physics.mcgill.ca

The proper time used in the Schwinger representation is the same as the worldsheet proper time τ used in the string theory. One might therefore expect Schwinger's proper-time representation to lead to string-like formulas in field theory, and that is indeed the case. The formalism of a string theory is very different from that of a field theory. In a critical string theory, every dynamical variable, be it spacetime, color, spin, hypercharge, or isospin, is treated on an equal footing and each is considered as a function of the worldsheet variables σ and τ. Reparametrization and conformal invariances of the theory give rise to independence of the scattering amplitudes on the geometry of the worldsheet. This in turn leads to duality, 'interactions without interactions', and other magical properties of the string; the low energy limit of the string theory gives automatically the Maxwell-Einstein-Yang-Mills gauge theory. In contrast, ordinary field theory treats the internal variables like color, spin, and isospin as fields of the spacetime variables. This asymmetry makes it complicated to enforce *local* gauge invariance, and produces enormous algebraic complications in actual calculations. A string-like formalism of field theory can therefore lead to simplifications and new understandings of gauge theory. Conversely, field theory is at ease with multiloop amplitudes with or without external fermions whereas string theory is not, so the latter may learn a trick or two from the former also. In short, the Schwinger representation is useful not only because it allows the spinor helicity technique to be applied to loop processes, but it also builds a bridge between the vastly different domains of field theory and string theory, thereby offering an avenue for cross fertilization.

A quick introduction to the spinor helicity technique for tree diagrams will be given in the next section, and a brief summary of loop diagrams in the Schwinger representation will be found in Sec. 3.

2. Spinor Helicity Technique for Tree Diagrams

Spin *is* an essential complication in actual calculation: if relativistic spin-$\frac{1}{2}$ particles are present, we must deal with Dirac matrices and hence four-channel problems, for example. This leads to a fair amount of algebraic complications that fortunately can be sidestepped at high energies, when the masses of the fermions and other external particles can be neglected. The method to accomplish that is the *spinor helicity technique*.

The physical basis of the simplification is as follows. For gauge theories, chirality is conserved at the vertices but it is helicity that is conserved along the propagators. The mixing of the chirality and helicity eigenstates along the way produces the four-channel problem. For massless fermions, chirality and helicity are identical, they do not change at all from beginning to end, thus a one-channel problem results when the fermion masses can be ignored.

To be more specific, let the massless Dirac wave functions be $u_\pm(p) = \pm v_\mp(p) \equiv |p\pm\rangle$ and $\bar{u}_\pm(p) = \pm\bar{v}_\mp(p) \equiv \langle p\pm|$. Chirality conservation implies $\langle p_i \pm |p_j\pm\rangle = 0$, leaving behind only the *overlap amplitudes* $\langle p_i + |p_j-\rangle \equiv [p_i p_j]$ and $\langle p_i - |p_j+\rangle \equiv \langle p_i p_j\rangle$ which do not vanish. In fact,

$$\langle p_i p_j\rangle [p_j p_i] = 2 p_i \cdot p_j, \tag{1}$$

and to within a phase factor, both $[p_ip_j]$ and $\langle p_ip_j\rangle$ are equal to $\sqrt{2p_i\cdot p_j}$.

Unlike massless fermions, gluon helicities are not conserved, but even so something can be said about the flow of their helicities. This is possible because kinematically a spin-1 particle can be regarded as a composite of two spin-$\frac{1}{2}$ particles. However, on account of gauge freedom the longitudinal polarization component is free, so the said composition cannot be unique. Mathematically, this is reflected in the following representation for the polarization vector $\epsilon_\mu^\pm$:

$$\epsilon_\mu^\pm(p,k) = \frac{\langle p\pm|\gamma_\mu|k\pm\rangle}{\sqrt{2}\langle k\mp|p\pm\rangle}, \tag{2}$$

where p is the photon/gluon momentum, and k is an *arbitrary* massless momentum called the *reference momentum*; different choices of k correspond to different choices of its gauge.

To see how this helps to visualize the flow of gluon/photon helicity, and to see how mathematically the four-channel fermion problem can now be reduced to a one-channel problem, we need two simple mathematical relations. The first is the completeness relation for massless fermions,

$$\gamma p_i = |p_i+\rangle\langle p_i+| + |p_i-\rangle\langle p_i-|, \tag{3}$$

and the second is the Fierz identity which reads

$$\begin{aligned}\langle A+|\gamma^\mu|B+\rangle\langle C-|\gamma_\mu|D-\rangle &= 2\langle A+|D-\rangle\langle C-|B+\rangle\\ \langle A+|\gamma^\mu|B+\rangle\langle C+|\gamma_\mu|D+\rangle &= 2\langle A+|C-\rangle\langle D-|B+\rangle.\end{aligned} \tag{4}$$

We can now understand how the γ-matrices can be eliminated in *tree diagrams*. Every internal momentum q_r of a tree can be written uniquely as a linear combination of the external massless momenta p_i,

$$q_r = \sum_i c_{ir}p_i \quad (\textbf{trees}). \tag{5}$$

Using (5) and (3), one can get rid of the γ-matrices in the propagators. Using (4) and (2), one can eliminate the γ-matrices at the vertices. The disappearance of the γ-matrices means that the four-channel problem is now reduced to a one-channel problem. The resulting scattering amplitude is a rational function of the overlap amplitudes $[p_ip_j]$ and $\langle p_ip_j\rangle$.

Moreover, one can devise a set of graphical rules to write down this rational function directly from the Feynman diagram[7]. To do so, each photon/gluon line is represented by a pair of fermion lines, and the resulting fermion lines are all connected continuously, allowed to terminate only at the external particles of the diagram. To each fermion line terminating with momenta p_i and p_j is associated an overlap amplitude factor $\langle p_i\pm|p_j\mp\rangle$. Fairly obvious rules are available to choose the polarization here. The numerator of the scattering amplitude is given, up to coupling constants etc., by the product of these factors. The denominators can also be converted into these factors by using (1). For example, the QED diagram Fig. 1(a) should be redrawn as Fig. 1(b) for this purpose. The p_i's are

the particle momenta and the k_i's are the reference momenta. The signs following the momenta symbols indicate the helicities. The numerator of the amplitude can be read off from Fig. 1(b) to be

$$S_0 = e^4 \cdot \langle p_3 q_2\rangle [q_2 k_6] \cdot \langle p_6 p_2\rangle \cdot [p_4 p_5] \cdot \langle k_5 q_1\rangle [q_1 p_1], \tag{6}$$

where overlap amplitudes involving off-shell momenta are defined with the help of (5), *e.g.,*

$$\langle p_3 q_2\rangle [q_2 k_6] \equiv \langle p_3, p_2 - p_6\rangle [p_2 - p_6, k_6] \equiv \langle p_3 p_2\rangle [p_2 k_6] - \langle p_3 p_6\rangle [p_6 k_6]. \tag{7}$$

If one looks at Fig. 1(b) carefully, one sees that the fermion lines sometimes turns one way at a vertex, but at other times turns the other way. This is all determined by which of the formulas in (4) one is using. The rule is as follows. Each fermion line has a fixed helicity. If one moves from one line to the other via a photon, then one continues along the original direction (opposite direction) if the second fermion line has the opposite (same) helicity as the first. When it comes to an external photon line with outgoing momentum p, reference momentum k, and helicity λ, then the rule is as follows. Imagine a fictitious external fermion line with an initial momentum k, a final momentum p, and helicity λ to be attached to the end of this photon line. Then one can use the rule devised above to proceed between the fermion lines.

Similar graphical rules can be devised for QCD, where one must take into account color flows in additional to the spin flows discussed here.

3. Multiloop Diagrams

Eq. (3) is violated for loop diagrams because of the presence of loop momenta. To enable the spinor helicity technique to be used one must first get rid of the loop momenta by integrating them out. This can indeed be accomplished in the Schwinger-parameter and the Feynman-parameter representations.

In the Schwinger representation, a proper-time parameter α is introduced for each internal line of momentum q to convert the denominator of every propagator to

$$\frac{1}{-q^2 + m^2 - i\epsilon} = i \int_0^\infty d\alpha \exp[-i\alpha(m^2 - q^2)]. \tag{8}$$

Suppose the Feynman diagram in question has ℓ loops with loop momenta k_a, n external lines with outgoing momenta p_i, and N internal lines with momenta q_r. Its scattering amplitude is given by

$$T(p) = \left[\frac{-i}{(2\pi)^4}\right]^\ell \int \prod_{a=1}^{\ell} (d^4 k_a) \frac{S_0(q,p)}{\prod_{r=1}^{N}(-q_r^2 + m_r^2 - i\epsilon)}, \tag{9}$$

where $S_0(q,p)$ consists of the vertices, the numerators of propagators, and possibly other coefficients. Substituting (8) into (9), the loop integrations can be carried out, leaving behind the Schwinger-parameter representation[8]

$$T(p) = \int_0^\infty [D\alpha]\ S(q,p)\exp\left[-iM + iP(\alpha,p)\right], \tag{10}$$

where

$$\begin{aligned} [D\alpha] &= \frac{i^N \Delta(\alpha)^{-2}}{(-16\pi^2)^\ell}\prod_{r=1}^N d\alpha_r \\ S(q,p) &= \sum_{k=0} S_k(q,p) \\ M &= \sum_r \alpha_r m_r^2 \\ P &= \sum_r \alpha_r q_r^2. \end{aligned} \tag{11}$$

There are three points about this new representation to keep in mind. First, correspondences with the Feynman diagram are maintained, though loop integrations are now replaced by $[D\alpha]$, S_0 is replaced by S, and the denominators of the propagators $(-q_r^2 + m_r^2 - i\epsilon)^{-1}$ are replaced by the exponentials $\exp[-i\alpha_r(-q_r^2 + m_r^2)]$. Secondly, the q_r in (10) is defined to be the current flowing through the rth internal line when the Feynman diagram is regarded as an electric circuit with resistances α_r and external currents p_i. It is no longer the same q_r as in (9); k_a is no longer present and (5) is restored, though c_{ir} is now dependent on the α's. Spinor helicity technique can once againn be applied and a concrete example will be discussed later. Note that the quantity P in (9) and (10) is just the power consumed by the circuit. As such, it is a quadratic form in p_i with coefficients given by the impedance matrix elements,

$$P = \sum_{i,j} Z_{ij} p_i \cdot p_j, \tag{12}$$

though it is important to note that because of momentum conservation, $\sum_i p_i = 0$, P is unchanged under a *level transformation*

$$Z_{ij} \to Z_{ij} + \xi_i + \xi_j \tag{13}$$

so the impedance matrix is not uniquely defined. In many respects this level transformation resembles a gauge transformation. Physical quantities such as the power P and the currents q_r are not altered by this transformation, but voltage levels at the vertices do change by a *common* amount under (13), which is why the transformation is so named. Measurable quantities such as voltage differences are not altered so neither are P nor q_r. As far as (10)–(12) are concerned, any *level scheme* (choice of ξ_i in (13)) of Z_{ij} will give identical

results. Later on, we have occasion to see formulas true only in particular level schemes. Thirdly, $S = S_0+S_1+S_2+\cdots$ contains the additional terms S_k $(k > 0)$, which are obtained from the original S_0 by contracting k pairs of q's via the rule

$$q_r^\mu q_s^\nu \to -\frac{i}{2} g^{\mu\nu} H_{rs}(\alpha). \tag{14}$$

If S_0 is a polynomial in the q_r's, then $S_k = 0$ for k larger than half of its degree, so the sum in S is a finite sum.

Simple rules can be derived with the help of graph theory to compute the electric circuit quantities like current q_r and power P, as well as the *Jacobian* $\Delta(\alpha)$ and the *contraction functions* $H_{rs}(\alpha)$:

$$\begin{aligned}
\Delta(\alpha) &= \sum_{T_1}(\prod^{\ell} \alpha),\\
\Delta\cdot P(\alpha,p) &= \sum_{T_2}(\prod^{\ell+1}\alpha)(\sum_1 p)^2,\\
-2\Delta\cdot Z_{ij} &= \sum_{T_2^{ij}}(\prod^{\ell+1}\alpha), \quad (\text{zero}-\text{diagonal level scheme}),\\
\Delta\cdot q_r &= \sum_{T_2(r)} \alpha_r^{-1}(\prod^{\ell+1}\alpha)(\sum_1 p),\\
\Delta\cdot H_{rr} &= -\partial\Delta(\alpha)/\partial\alpha_r,\\
\Delta\cdot H_{rs} &= \pm\sum_{T_2(rs)}(\alpha_r\alpha_s)^{-1}(\prod^{\ell+1}\alpha), \quad (r\neq s).
\end{aligned} \tag{15}$$

These formulas should be interpreted as follows. An ℓ-loop diagram can be changed into a tree by cutting ℓ lines, and into two disconnected tress (a *'2-tree'*) by cutting $\ell+1$ lines. The sums in (15) are taken over the collection T_1 of all such trees in the case of Δ, over the collection T_2 of all such 2-trees in the case of P, and over the collection $T_2(r)$ of all 2-trees in which the line r must be cut to form them, in the case of q_r. For H_{rs}, the sum is over the collection $T_2(rs)$ of 2-trees in which lines r and s must be cut, and such that a single tree results if either of them is inserted back. For Z_{ij}, where the formula is true only in the *zero-diagonal level scheme* in which $Z_{ii} = 0$ for all i, the sum is taken over the set T_2^{ij} of 2-trees in which vertices i and j belong to separate trees. In all cases, $\prod\alpha$ indicates the product of the α's of the cut lines, and $\sum_1 p$ denotes the sum of external momenta attached to either one of the two trees. The signs involved in the formulas for q_r and H_{rs} can also be determined easily.

For example, (15) leads to the following results for the circuit in Fig. 2.

$$\Delta = (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+\alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$

$$
\begin{aligned}
-2\Delta \cdot Z_{13} &= (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4) + \alpha_5(\alpha_1+\alpha_4)(\alpha_2+\alpha_3) \\
\Delta \cdot q_3 &= \alpha_5\alpha_1(p_1+p_2) + (\alpha_5\alpha_2)p_2 - \alpha_4(\alpha_1+\alpha_2+\alpha_5)p_3 \\
\Delta \cdot P &= \alpha_1\alpha_5\alpha_3(p_1+p_2)^2 + \alpha_2\alpha_5\alpha_4(p_1+p_4)^2 + \alpha_1\alpha_2(\alpha_3+\alpha_4+\alpha_5)p_1^2 + \\
&\quad + \alpha_3\alpha_4(\alpha_1+\alpha_2+\alpha_5)p_3^2 + \alpha_1\alpha_5\alpha_4 p_4^2 + \alpha_2\alpha_5\alpha_3 p_2^2
\end{aligned}
$$

The spin flow for the numerator S_0 in (10) can be read off directly from the Feynman diagram as before. For example, Fig. 3(a) should first be drawn like Fig. 3(b), from which one gets immediately that

$$S_0 = e^4 \cdot \langle p_4 q_2\rangle [q_2 p_3] \cdot [p_2 q_1]\langle q_1 q_4\rangle [q_4 q_3]\langle q_3 p_1\rangle.$$

Similar rules can be devised for other S_k. In the case of QCD, there are also analogous graphical rules for color flows.

The integrand of the scattering amplitude in (9) is a function of α and p, but it is not just any function of them. The integrand is composed of electric circuit quantities. As such, they obey the Kirchhoff laws and a set of differential identities

$$
\begin{aligned}
\frac{\partial P}{\partial \alpha_s} &= \frac{\partial}{\partial \alpha_s}\left(\sum_r \alpha_r q_r^2\right) = q_s^2, \\
\frac{\partial q_r}{\partial \alpha_s} &= H_{rs} q_s, \\
\frac{\partial H_{rs}}{\partial \alpha_t} &= H_{rt} H_{ts}.
\end{aligned}
\tag{16}
$$

Many other identities can be derived from these.

These identities can be used to reshape (9) into other expressions. In particular, into a string-like form. We shall do that only for scalar electrodynamics but similar formulas are known for QED and QCD. Consider first a one-loop n-photon amplitude, given by Fig. 4(a). In scalar electrodynamics, photons are derivatively coupled to the charged particles, hence

$$S_0(q,p) = \prod_a [e\epsilon_a \cdot (q_{a'} + q_{a''})] \equiv S_0^{ext}(q,p). \tag{17}$$

Using (16), one can transform the integrand of (9) into a form obtained in string theory[5] and the first-quantized formalism[6]:

$$S(q,p)\exp[-i(M-P)] = \exp[-i(M-P')]_{ml}, \tag{18}$$

where

$$P' = \sum_{a,b}(p_a - ie\epsilon_a\partial_a)\cdot(p_b - ie\epsilon_b\partial_b)Z_{ab}, \tag{19}$$

with $\partial_a \equiv \partial/\partial\alpha_{a'} - \partial/\partial\alpha_{a''}$, if $P = \sum_{a,b} p_a \cdot p_b Z_{ab}$. Eq. (18) is true only in the zero-diagonal level scheme where $Z_{aa} = 0$ for every vertex a. The subscript ml instructs

us to expand the exponential and keep only the terms multilinear in all the ϵ_a's. For arbitrary multiloop processes like Fig. 4(b), a similar string-like formula exists[7]. In this case $S_0(q,p) = S_0^{ext}(q,p)S_0^{int}(q,p)$, where S_0^{ext} is the product of vertex factors at vertices with an external photon line, as in (17), and S_0^{int} is the rest of the vertices, indicated by heavy dots in Fig. 4(b). Eq. (18) now takes on the form

$$S(q,p)\exp[-i(M-P)] = S^{int}(q',p)\exp[-i(M-P')]_{ml}, \tag{20}$$

for some suitably defined q', the detail of which is discussed in Ref. 7.

References

1. F.A. Berends, R. Kleiss, P. De Causmaeker, R. Gastmans, and T.T. Wu, *Phys. Lett.* 103B (1981), 124; P. De Causmaecker, R. Gastmans, W. Troost and T.T. Wu, *Phys. Lett.* 105B (1981), 215.
2. Z. Xu, D.H. Zhang, and L. Chang, *Tsinghua Univesity preprints* 84/4,5,6 (1984); *Nucl. Phys.* B291 (1987), 392.
3. S. Parke and T. Taylor, *Phys. Rev. Lett.* 56 (1986), 2459.
4. M.L. Mangano and S.J. Parke, *Phys. Rep.* 200 (1991), 301; R. Gastmans and T.T. Wu, 'The Ubiquitous Photon', International Series of Monographs on Physics, Vol. 80 (Clarendon Press, Oxford, 1990).
5. Z. Bern and D.K. Kosower, *Phys. Rev. Lett.* B86 (1991), 1669.
6. M. Strassler, *Nucl. Phys.* B385 (1992), 145.
7. C.S. Lam, *Nucl. Phys.* B397 (1993), 143; *Phys. Rev.* **D48** (1993), 873; *Can. J. Phys.* (1994) (hep-ph/9308289).
8. C.S. Lam and J.P. Lebrun, *Nuovo Cimento* 59A (1969), 397.